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NONLINEAR EIGENVALUE PROBLEMS ON INFINITE INTERVALS

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ABSTRACT

This paper is concerned with nonlinear eigenvalue problems of boundary value problems for ordinary differential equation posed on an infinite interval. It is shown that under certain analyticity assumptions - a domain in the complex plain can be identified, in which all eigenvalues are isolated. An intriguing way to solve such problems is to cut the infinite interval at a finite but large enough point and to impose additional, so called asymptotic boundary conditions at this far end. The obtained eigenvalue problem for the two point boundary value problem on this finite but large interval can be solved by any appropriate code. In this paper suitable asymptotic boundary conditions are devised and the order of convergence, as the length of the interval, on which these approximating problems are posed, converges to infinity, is investigated. Exponential convergence is shown for well posed approximating problems.

AMS (MOS) Subject Classifications: 34B25, 34D05, 34B05, 34P30

Key Words: Spectral theory of boundary value problems, Asymptotic properties, Asymptotic expansion, Nonlinear eigenvalue problems

Work Unit Number 3 (Numerical Analysis and Computer Science)

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SIGNIFICANCE AND EXPLANATION

This paper deals with the analysis and approximation theory of nonlinear eigenvalue problems for boundary value problems of ordinary differential equation posed on an infinite interval. These problems have the following forms. We have a linear homogeneous system of ordinary differential equation depending nonlinearly on a complex parameter λ. This system is defined on the interval $[t_0,\infty]$ where t_0 is some finite point. Moreover we have linear homogenous boundary conditions at t_0 and a continuity requirement at infinity. We look for values of the parameter λ so that a function $y \not\equiv 0$ exists which fulfills the system of differential equation over the whole interval $[t_0,\infty)$, the boundary conditions at t_0 and the continuity requirement at infinity.

Problems of this kind frequently occur in fluid mechanics when the stability of a laminar boundary layer is investigated (Orr-Sommerfeld problem) and in quantum physics.

In this paper properties of the eigenvalues λ and the eigenfunctions y are investigated.

An intriguing way to solve such problems is to cut the infinite interval at a finite but large enough point T and to impose additional - so called asymptotic - boundary conditions at T, which substitute the continuity requirement at infinity. The resulting finite eigenvalue problem can be solved by any appropriate code, for example by collocation methods.

Suitable asymptotic boundary conditions are derived and the convergence of eigenvalues and eigenfunctions as the length of the interval, on which the finite approximating problems are posed, converges to infinity, is investigated. Exponential convergence is shown for the most important cases.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NONLINEAR EIGENVALUE PROBLEMS ON INFINITE INTERVALS

Peter A. Markowich and Richard Weiss*

1. INTRODUCTION.

This paper is concerned with nonlinear eigenvalue problems of the following form

$$y' = t^{\alpha}A(t,\lambda)y, \quad 1 \leq t < \infty, \quad \alpha > -1 \tag{1.1}$$

$$B(\lambda)y(1) = 0 \tag{1.2}$$

$$y \in C([1,\infty]) :\Leftrightarrow y \in C([1,\infty)) \text{ and } \lim_{t\to\infty} y(t) \text{ exists} \tag{1.3}$$

where y is an n-vector and $A(t,\lambda)$ is an $n \times n$ matrix. Equation (1.1) has a singularity of the second kind of rank $\alpha + 1$ in $t = \infty$.

A solution of (1.1), (1.2), (1.3) is given by a pair (μ,y), $\mu \in C$ such that $y \not\equiv 0$ fulfills (1.1), (1.2) with $\lambda = \mu$ and (1.3). Eigenvalue problems on infinite intervals occur frequently in quantum mechanics and in fluid mechanics, when the stability of laminar flows over infinite media is investigated (see Ng and Reid (1980)).

de Hoog and Weiss (1980a) and Markowich (1980a) treated linear eigenvalue problems on infinite intervals, i.e. $A(t,\lambda) = A_0(t) + \lambda A_1(t)$ and $A_0, A_1 \in C([1,\infty])$ and $B(\lambda) \equiv B$. It was shown that all eigenvalues λ of this linear eigenvalue problem, for which the matrix $A(\infty,\lambda) = A_0(\infty) + \lambda A_1(\infty)$ has no eigenvalue on the imaginary axis, are isolated, if not all $\lambda \in C$ are eigenvalues. Moreover if $A_1(\infty) = 0$ there is an infinite sequence of eigenvalues λ_i which fulfill $|\lambda_i| \to \infty$. de Hoog and Weiss (1980a) also proved that the spectral subspces are finite dimensional.

The first goal of this paper is to show the generalization of the isolatedness-statement to nonlinear eigenvalue problems assuming that $B(\lambda)$, $A(t,\lambda)$ are analytic in $\lambda \in \phi \supset \Omega$, where Ω is the domain in which $A(\infty,\lambda)$ has no eigenvalue on the imaginary

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axis. The analyticity is supposed to hold for all $t \in [1,\infty]$ and $A(t,\lambda)$ is jointly continuous in $[1,\infty] \times \Omega$. The number of rows of the matrix $B(\lambda)$ is assumed to equal r_-, which is the sum of algebraic multiplicities of eigenvalues of $A(\infty,\lambda)$ with negative real part for $\lambda \in \Omega$.

The second goal of this paper is to investigate the approximating eigenvalue problems

$$x_T' = t^\alpha A(t,\lambda)x_T, \quad 1 < t < T, \quad T >> 1 \tag{1.3}$$

$$B(\lambda)x_T(1) = 0 \tag{1.4}$$

$$S(\lambda)x_T(T) = 0 \tag{1.5}$$

where $S(\lambda)$ is a suitably chosen matrix with $r_+ = n - r_-$ rows.

The main question arising here is to find out which matrices $S(\lambda)$ lead to convergence of the eigenvalues and eigenfunctions of these approximating problems to the eigenvalues and eigenfunctions of (1.1), (1.2), (1.3) as $T \to \infty$. A class of matrices $S(\lambda)$ which implies exponential convergence will be identified. The convergence results are the generalization of the results obtained by Markowich (1980a) for linear eigenvalue problems. As Markowich (1980a) pointed out there is not always (even in the case of a linear eigenvalue problem) an obvious way to choose the suitable S which is independent of λ. However there is an intrinsic way (see Keller (1976)) to set up an 'asymptotic' boundary condition S depending on λ. Therefore these 'finite' eigenvalue problems are, even in the case of a linear 'infinite' problem, nonlinear.

This paper is organized as follows, in Chapter 2 nonlinear finite dimensional eigenvalue problems are discussed, Chapter 3 is concerned with the case when A is independent of t, in Chapter 4 this restriction is dropped and Chapter 5 is concerned with examples illustrating the theory.

2. FINITE DIMENSIONAL NONLINEAR EIGENVALUE PROBLEMS.

Let $A(\lambda)$ be a $k \times k$ matrix, holomorphic in some domain $\Omega \subset \mathbb{C}$. A value $\mu \in \mathbb{C}$ for which the linear equation

$$A(\mu)\xi = 0, \quad \xi \neq 0 \tag{2.1}$$

possesses a solution, is called an eigenvalue and ξ is a corresponding eigenvector. Let $\det A(\lambda)$ denote the determinant of $A(\lambda)$. Since (2.1) holds iff $\det A(\mu) = 0$ it follows from the identity theorem of holomorphic functions that either all $\lambda \in \Omega$ are eigenvalues or every compact subset of Ω contains at most finitely many eigenvalues.

Let $\varepsilon \in (0,\varepsilon_0]$ be a real parameter and $B(\lambda,\varepsilon)$ be a $k \times k$ matrix, holomorphic in Ω for all $\varepsilon \in (0,\varepsilon_0]$ and fulfilling

$$\lim_{\varepsilon \to 0} \sup_{\lambda \in \Lambda} \|B(\lambda,\varepsilon)\| = 0 \quad \text{for all } \Lambda \text{ compact}, \ \Lambda \subset \Omega \tag{2.2}$$

where $\|\cdot\|$ denotes some matrix norm. Now consider the perturbed nonlinear eigenvalue problem

$$C(\lambda,\varepsilon)\xi \equiv (A(\lambda) + B(\lambda,\varepsilon))\xi = 0, \quad \xi \neq 0 \tag{2.3}$$

Since

$$\lim_{\varepsilon \to 0} \det C(\lambda,\varepsilon) = \det A(\lambda) \tag{2.4}$$

we may employ standard perturbation results for zeros of holomorphic functions. Therefore, let μ be a root of order s of $\det A(\lambda) = 0$, θ be a neighbourhood of μ and

$$b(\varepsilon) = \sup_{\lambda \in \theta} |\det A(\lambda) - \det C(\lambda,\varepsilon)| \tag{2.5}$$

Then we get

Theorem 2.1.

(i) When ε is sufficiently small there are precisely s eigenvalues $\mu_\varepsilon^1,\dots,\mu_\varepsilon^s$ of (2.3) near μ (counting multiplicities) and they satisfy

$$|\mu_\varepsilon^j - \mu| \leq \text{const.}\, b(\varepsilon)^{\frac{1}{s}}, \quad j = 1,\dots,s \tag{2.6}$$

(ii) The mean

$$\mu_\varepsilon = \frac{1}{s}\sum_{i=1}^{s} \mu_\varepsilon^i \tag{2.7}$$

fulfills

$$|\mu_\varepsilon - \mu| \leq \text{const.}b(\varepsilon) \, . \tag{2.8}$$

The perturbation statement for the eigenvectors is weaker.

Theorem 2.2. Let $\varepsilon_n \to 0$ as $n \to \infty$ and let ξ_{ε_n} be a sequence of eigenvectors of (2.3) (with norm one) each of them belonging to a $\mu_{\varepsilon_n}^i$ for $i = 1,\ldots,s$, then there is a subsequence $\xi_{\varepsilon_{n_k}}$ which converges to an eigenvector ξ with norm one of (2.1) and

$$\inf_{\xi \in N(A(\mu))} \|\xi - \xi_{\varepsilon_n}\| \leq \text{const.}b(\varepsilon_n)^{\frac{1}{s}} \tag{2.9}$$

$N(A(\mu))$ denotes the nullspace. A proof can be found in G. Vainniko (1976), par. 4.

In the case of a linear eigenvalue problem $A(\lambda) \equiv A - \lambda I$ we get a stronger perturbation result for eigenvectors if the algebraic and geometric multiplicity of the eigenvalue μ is equal to one. Therefore we define:

Definition 2.1. The eigenvalue μ of (2.1) is called simple if μ is a zero of order one of $\det A(\lambda) = 0$.

It is easily seen that μ is a simple eigenvalue of (2.1) iff $\det(A(\mu) + \tau A'(\mu)) = 0$ has a zero of order one at $\tau = 0$ which holds iff $\tau = 0$ is an eigenvalue of geometric and algebraic multiplicity 1 of the generalized linear eigenvalue problems

$$(A(\mu) + \tau A'(\mu))\xi = 0 \, . \tag{2.10}$$

We get

Theorem 2.3. Let μ be a simple eigenvalue of (2.1). Then:

(i) for ε sufficiently small there is a unique eigenvalue μ_ε of (2.3) and it satisfies

$$|\mu_\varepsilon - \mu| \leq \text{const} \, |\det A(\mu) - \det C(\mu,\varepsilon)| \tag{2.11}$$

(ii) for every μ_ε there is exactly one eigenvector ξ_ε (with norm one) of (2.3) which satisfies

$$\|\xi_\varepsilon - \xi\| \leq \text{const} \, |\det A(\mu) - \det C(\mu,\varepsilon)| \, . \tag{2.12}$$

Proof. The equations

$$C(\lambda,\varepsilon)\xi_\varepsilon = 0, \quad \|\xi_\varepsilon\| = 1 \tag{2.13}$$

where $\|\cdot\|$ indicates the euclidean norm in $\mathbf{C}^k$ are a nonlinear system of equations for $(\mu_\varepsilon,\xi_\varepsilon)$. The Frechet derivative of the unperturbed problem $(\varepsilon = 0)$ at (μ,ξ) is given by the matrix

$$\begin{bmatrix} A(\mu) & A'(\mu)\xi \\ \xi^T & 0 \end{bmatrix} \tag{2.14}$$

which is nonsingular because μ is simple. (i) and (ii) follow in a straight forward way by applying the techniques of Keller (1975) and Vainniko (1976) par. 4. Here θ shrinks to the point μ.

We conclude this section with a result on holomorphic families of projections.

Theorem 2.4. Let $P(\lambda) : \mathbf{C}^k \to \mathbf{C}^k$ be a family of projections holomorphic for $\lambda \in \Omega$. Then

(i) for any pair $(\lambda_1,\lambda_2) \in \Omega \times \Omega$ there is a nonsingular $n \times n$ matrix $Q(\lambda_1,\lambda_2)$ such that

$$P(\lambda_1) = Q(\lambda_1,\lambda_2)^{-1}P(\lambda_2)Q(\lambda_1,\lambda_2) \tag{2.15}$$

(ii) $P(\lambda_1)\mathbf{C}^k$ is isomorphic to $P(\lambda_2)\mathbf{C}^k$ for all λ_1,λ_2 Ω.

(iii) Let $r = \operatorname{rank} P(\lambda)$. Then there is a $k \times k$ matrix of rank r, holomorphic in Ω, whose columns span $P(\lambda)\mathbf{C}^k$.

Proof. (i) follows from Kato (1966) and (ii), (iii) follow easily from (i).

3. NONLINEAR CONSTANT - COEFFICIENT EIGENVALUE PROBLEMS.

We consider

$$y' = t^{\alpha}A(\lambda)y, \quad 1 \leq t < \infty, \quad \alpha > -1 \tag{3.1}$$

$$B(\lambda)y(1) = 0 \tag{3.2}$$

$$y \in C([1,\infty]) \tag{3.3}$$

where $A(\lambda) \neq 0$ is an $n \times n$ matrix.

The analysis for these problems will outline the approach for the more complicated case, when A is also a function of the independent variable t. We assume that $A(\cdot)$, $B(\cdot)$ are holomorphic in some domain ϕ in the complex plain and that there is a domain $\Omega \subset \phi$, so that $A(\lambda)$ has no eigenvalue $\nu(\lambda)$ with vanishing real part. Therefore $A(\lambda)$ for all $\lambda \in \Omega$ has a fixed number of eigenvalues with a negative real part, which we call r_- and a fixed number of eigenvalues with a positive real part, which we call r_+ $(r_+ + r_- = n)$. Now we take a compact subset $\Lambda \subset \Omega$. Then there are two closed rectifiable curves Γ_+, Γ_-, completely in the right resp. left half plain, so that for all $\lambda \in \Lambda$ all eigenvalues of $A(\lambda)$ are enclosed by either Γ_+ or Γ_-.

Now let

$$P_+(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_+} (z - A(\lambda))^{-1}dz, \quad \text{rank}P_+(\lambda) = r_+ \tag{3.4}$$

$$P_-(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_-} (z - A(\lambda))^{-1}dz, \quad \text{rank}P_-(\lambda) = r_- . \tag{3.5}$$

be the total projections onto the direct sum of invariant subspaces associated with eigenvalues of $A(\lambda)$ with positive resp. negative real part.

From Kato (1966), Chapter 2 we conclude that P_+, P_- are holomorphic in Λ°, the interior of Λ.

The general solution of the problem (3.1), (3.3) is

$$y(t,\lambda) = \exp\left(\frac{t^{\alpha+1}}{\alpha + 1} A(\lambda)\right)P_-(\lambda)\xi, \quad \xi \in C^n \tag{3.6}$$

Theorem 2.4, (iii), implies that there is a $n \times r_-$ matrix $V(\lambda)$ of full rank and holomorphic in Λ° which spans $P_-(\lambda)\mathbf{C}^n$. Therefore we can rewrite (3.6) and get, inserting into the boundary condition (3.2)

$$F(\lambda)\eta \equiv B(\lambda)\exp\left(\frac{A(\lambda)}{\alpha+1}\right) V(\lambda)\eta = 0, \quad \eta \in C^{r_-} \tag{3.7}$$

assuming that $B(\lambda)$ is a $r_- \times n$ matrix. Every pair (μ,η), $\eta \neq 0$ which solves (3.7) determines a solution of the eigenvalue problem (3.1), (3.2), (3.3) by

$$y(t,\mu) = \exp\left(\frac{t^{\alpha+1}}{\alpha+1} A(\mu)\right)V(\mu)\eta . \tag{3.8}$$

Our assumptions guarantee that $F(\lambda) \equiv B(\lambda)\exp\left(\frac{A(\lambda)}{\alpha+1}\right) V(\lambda)$ is holomorphic in Λ°, so we get from Chapter 2

<u>Theorem 3.1</u>. Let $B(\lambda)$ be a $r_- \times n$ matrix. Then either all $\lambda \in \Omega$ are eigenvalues of (3.1), (3.2), (3.3) or every compact subset of Ω contains at most a finite number of eigenvalues. If μ is an eigenvalue of (3.1), (3.2), (3.3), the dimension of the nullspace is between 1 and r_-.

Now we want to approximate the eigenvalue problem (3.1), (3.2), (3.3) by finite interval problems:

$$x_T' = t^\alpha A(\lambda)x_T, \quad 1 < t < T, \quad T \gg 1 \tag{3.9}$$

$$B(\lambda)x_T(1) = 0 \tag{3.10}$$

$$S(\lambda)x_T(T) = 0 \tag{3.11}$$

where $S(\lambda)$ is a $r_+ \times n$ matrix whose choice will be discussed later.

We write the general solution of (3.9) as

$$x_T(t) = \exp\left(\frac{t^{\alpha+1}}{\alpha+1} A(\lambda)\right)V(\lambda)\eta_- + \exp\left(\frac{t^{\alpha+1} - T^{\alpha+1}}{\alpha+1} A(\lambda)\right)W(\lambda)\eta_+ \tag{3.12}$$

where the columns of the $n \times r_+$ matrix $W(\lambda)$, which is holomophic in Λ°, span $P_+(\lambda)\mathbf{C}^n$. Evaluation at the boundaries and using (3.10), (3.11) gives the $n \times n$ block system

$$F(\lambda,T)\binom{\eta_-}{\eta_+} = \begin{bmatrix} B(\lambda)\exp\left(\frac{A(\lambda)}{\alpha+1}\right)V(\lambda) & B(\lambda)\exp\left(\frac{1-T^{\alpha-1}}{\alpha+1}A(\lambda)\right)W(\lambda) \\ S(\lambda)\exp\left(\frac{T^{\alpha+1}}{\alpha+1}A(\lambda)\right)V(\lambda) & S(\lambda)W(\lambda) \end{bmatrix}\binom{\eta_-}{\eta_+} = 0 \qquad (3.13)$$

We conclude that

$$\det F(\lambda,T) = \det F(\lambda)\cdot\det S(\lambda)W(\lambda) + c(\lambda,T) \qquad (3.14)$$

holds, where

$$|c(\lambda,T)| \leq \text{const.}\left\|\exp\left(\frac{-T^{\alpha+1}}{\alpha+1}A(\lambda)\right)W(\lambda)\right\|\cdot\left\|S(\lambda)\exp\left(\frac{T^{\alpha+1}}{\alpha+1}A(\lambda)\right)V(\lambda)\right\| \qquad (3.15)$$

Let $\nu_-(\lambda)$ be the largest negative real part of the eigenvalues of $A(\lambda)$ and let $\nu_+(\lambda)$ be the smallest positive real part of the eigenvalues. Then (3.15) reduces to

$$|c(\lambda,T)| \leq \text{const}(\lambda)\exp\left((\nu_-(\lambda) - \nu_+(\lambda))\frac{T^{\alpha+1}}{\alpha+1}\right) \qquad (3.16)$$

where $\text{const}(\lambda)$ is bounded when λ varies in a compact set. Now we prove the convergence theorem.

<u>Theorem 3.2</u>: Let the $r_+ \times n$ matrix $S(\lambda)$ be holomorphic for $\lambda \in \Omega$ and assume that $\det S(\lambda)W(\lambda) \neq 0$ for $\lambda \in \Omega$. Let $\mu \in \Omega$ be an eigenvalue of (3.1), (3.2), (3.3) of order s, i.e. $\det F(\lambda)$ has a zero at $\lambda = \mu$ of order s. Then there are exactly s eigenvalues $\mu_T^1,\ldots,\mu_T^s$ (counting multiplicities of the zeros of $\det F(\lambda,T)$) for T sufficiently large in a sufficiently small neighbourhood of μ and

$$\max_{i=1(1)s}|\mu_T^i - \mu| \leq \text{const.}\exp\left((\nu_-(\mu) - \nu_+(\mu) + \varepsilon)\frac{T^{\alpha+1}}{s(\alpha+1)}\right) \qquad (3.17)$$

$$|\mu_T - \mu| \leq \text{const.}\exp\left((\nu_-(\mu) - \nu_+(\mu) + \varepsilon)\frac{T^{\alpha+1}}{\alpha+1}\right) \qquad (3.18)$$

where $\mu_T = \frac{1}{s}\sum_{i=1}^{s}\mu_T^i$ and $\varepsilon > 0$ is sufficiently small. Let x_T be an eigenfunction belonging to one of the μ_T^i's. Then

$$\inf_{y\in N_\mu}\|x_T - y\|_{[1,T]} \leq \text{const.}\exp\left((\nu_-(\mu) - \nu_+(\mu) + \varepsilon)\frac{T^{\alpha+1}}{s(\alpha+1)}\right) \qquad (3.19)$$

where N_μ denotes the nullspace of (3.1), (3.2), (3.3) for $\lambda = \mu$.

Proof. All statements follow immediately by regarding (3.13) as perturbed eigenvalue problem of

$$\begin{bmatrix} B(\lambda)\exp\left(\frac{A(\lambda)}{\alpha+1}\right)V(\lambda) & 0 \\ 0 & S(\lambda)W(\lambda) \end{bmatrix} \binom{\eta}{\xi} = 0 \tag{3.20}$$

(which is equivalent to (3.7)) and by applying the Theorems 2.1 and 2.2.

Now we discuss a possible choice of $S(\lambda)$. Let the rows of the $r_+ \times n$-matrix $S_p(\lambda)$ span the range of $(P_+(\lambda))^T$ (the superscript denotes transposition). Then the asymptotic boundary condition

$$S_p(\lambda)x_T(T) = 0 \tag{3.21}$$

fulfills the assumptions of Theorem 3.2. Moreover

$$P_+(\lambda)\exp\left(\frac{T^{\alpha+1}}{\alpha+1} A(\lambda)\right)V(\lambda) \equiv 0, \quad \lambda \in \Omega \tag{3.22}$$

holds. Therefore, using the boundary condition (3.21), the matrix in the (2.1) position in (3.13) vanishes for all $T > 1$ and the approximate problems (3.9), (3.10), (3.11) reproduce the eigenvalues and eigenfunctions of the problem (3.1), (3.2), (3.3) exactly.

4. GENERAL NONLINEAR EIGENVALUE PROBLEMS ON INFINITE INTERVALS.

We consider the following problem

$$y' = t^{\alpha}A(t,\lambda)y, \quad 1 \le t < \infty, \quad \alpha > -1 \tag{4.1}$$

$$B(\lambda)y(1) = 0 \tag{4.2}$$

$$y \in C([1,\infty]) \tag{4.3}$$

where $A(t,\lambda)$ is an $n \times n$ matrix holomorphic for λ in some domain Φ and every fixed $t \in [1,\infty]$ and continuous in $[1,\infty] \times \theta$. Also B is holomorphic in Φ. We assume that there is a domain $\Omega \subset \Phi$, so that the matrix $A(\infty,\lambda)$ has no eigenvalue $\nu(\lambda)$ on the imaginary axis for $\lambda \in \Omega$. As in Chapter 3 we take any compact subset $\Lambda \subset \Omega$ and construct the projections $P_+(\lambda)$, $P_-(\lambda)$

$$P_+(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_+} (z - A(\infty,\lambda))^{-1} dz, \quad \mathrm{rank} P_+(\lambda) \equiv r_+ \tag{4.4}$$

$$P_-(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_-} (z - A(\infty,\lambda))^{-1} dz, \quad \mathrm{rank} P_-(\lambda) \equiv r_- . \tag{4.5}$$

The contours Γ_+, Γ_- are chosen as in Chapter 3. We set

$$\phi(t,\lambda) = \exp\left(\frac{A(\infty,\lambda)}{\alpha+1} t^{\alpha+1}\right) \tag{4.6}$$

and define the operator $H_\lambda : C([\delta,\infty]) \to C([\delta,\infty])$ for $\delta > 1$

$$\begin{aligned}(H_\lambda g)(t) &= \phi(t,\lambda) \int_\infty^t P_+(\lambda)\phi^{-1}(s,\lambda)s^\alpha g(s)ds + \\ &+ \phi(t,\lambda) \int_\delta^t P_-(\lambda)\phi^{-1}(s,\lambda)s^\alpha g(s)ds\end{aligned} \tag{4.7}$$

so that $H_\lambda g \in C([\delta,\infty])$ is a particular solution of the problem

$$\tilde{y}' = t^\alpha A(\infty,\lambda)\tilde{y} + t^\alpha g(t), \; t > \delta, \; g \in C([\delta,\infty]) . \tag{4.8}$$

An analysis of H_λ can be found in de Hoog and Weiss (1980a,b). P_+, P_- are holomorphic in Λ^o for every (fixed) $t \in [\delta,\infty]$ and continuous for $t \in [1,\infty]$. Then it is an easy exercise to show that $(H_\lambda g(\cdot,\lambda))(t)$ is holomorphic in Λ^o for every fixed $t \in [\delta,\infty]$. From de Hoog and Weiss (1980a,b) we conclude that

$$\|H_\lambda\|_{[\delta,\infty]} \le C(\lambda) \tag{4.9}$$

where $C(\lambda)$ is independent of δ.

Now we show that $C(\lambda)$ remains bounded when λ varies in compact subsets $K \subset \Lambda^o$. From (4.7) we derive

$$\|H_\lambda\|_{[\delta,\infty]} \le \max_{t\in[\delta,\infty]} \left(\int_t^\infty \|F_+(t,s,\lambda)\|ds + \int_\delta^t \|F_-(t,s,\lambda)\|ds\right) \tag{4.10}$$

where

$$\text{(a)} \quad F_+(t,s,\lambda) = s^{\alpha}\phi(t,\lambda)P_+(\lambda)\phi^{-1}(s,\lambda), \quad \lambda \in \Lambda$$

$$\text{(b)} \quad F_-(t,s,\lambda) = s^{\alpha}\phi(t,\lambda)P_-(\lambda)\phi^{-1}(s,\lambda), \quad \lambda \in \Lambda \tag{4.11}$$

holds. We transform $A(\infty,\lambda)$ to its Jordan canonical form $J(\infty,\lambda)$:

$$A(\infty,\lambda) = E(\lambda)J(\infty,\lambda)E^{-1}(\lambda), \quad \lambda \in \Omega \tag{4.12}$$

and assume that $J(\infty,\lambda)$ has the block structure

$$J(\infty,\lambda) = \operatorname{diag}(J_\infty^+(\lambda),J_\infty^-(\lambda)) \tag{4.13}$$

where the $r_+ \times r_+$ matrix $J_\infty^+(\lambda)$ contains only eigenvalues with positive real part and the $r_- \times r_-$ matrix $J_\infty^-(\lambda)$ contains only eigenvalues with negative real part for all $\lambda \in \Omega$. Defining the diagonal projection

$$D_+ = \operatorname{diag}(I_{r_+},0), \quad D_- = \operatorname{diag}(0,I_{r_-}) \tag{4.14}$$

we get

$$P_+(\lambda) = E(\lambda)D_+E^{-1}(\lambda), \quad P_-(\lambda) = E(\lambda)D_-E^{-1}(\lambda), \quad \lambda \in \Omega \tag{4.15}$$

and obtain

$$\text{(a)} \qquad F_+(t,s,\lambda) = s^{\alpha}E(\lambda)\begin{bmatrix} \exp\left(\frac{J_\infty^+(\lambda)}{\alpha+1}(T^{\alpha+1}-s^{\alpha+1})\right) & 0 \\ 0 & 0 \end{bmatrix} E^{-1}(\lambda)$$

$$\text{(b)} \qquad F_-(t,s,\lambda) = s^{\alpha}E(\lambda)\begin{bmatrix} 0 & 0 \\ 0 & \exp\left(\frac{J_\infty^-(\lambda)}{\alpha+1}(t^{\alpha+1}-s^{\alpha+1})\right) \end{bmatrix} E^{-1}(\lambda) \tag{4.16}$$

Obviously, $F_+(t,s,\cdot)$, $F_-(t,s,\cdot)$ are holomorphic in Λ° for fixed s,t.

Each entry of F_+, F_- is a sum of the form

$$f_\pm(t,s,\lambda) = s^{\alpha}\sum_{i=1}^{r_\pm} a_i(\lambda)\exp\left(\frac{\nu_i^\pm(\lambda)}{\alpha+1}(t^{\alpha+1}-s^{\alpha+1})\right)(t^{\alpha+1}-s^{\alpha+1})^{j_i} \tag{4.17}$$

where $\nu_i^+(\lambda)$ are the eigenvalues of $J_\infty^+(\lambda)$ and $\nu_i^-(\lambda)$ are the eigenvalues of $J_\infty^-(\lambda)$.

The integers j_i fulfills $0 \leq j_i \leq (r_+ - 1)$ resp. $0 \leq j_i \leq (r_- - 1)$. $a_i(\lambda)$ is a sum of products of elements of $E(\lambda)$ and $E^{-1}(\lambda)$.

Now we take a compact subset $K \subset \Lambda°$. It follows from Kato (1966) that $E(\lambda)$, $E^{-1}(\lambda)$ can be chosen holomorphically in $K - \{z_1,\dots,z_n\}$ where the z_i are points at which eigenvalues $\nu_i^{\pm}(\lambda)$ change algebraic or geometric multiplicities. Also $J(\lambda)$ is holomorphic in $K - \{z_1,\dots,z_n\}$. Without loss of generality we assume that none of the z_i's lies on the boundary ∂K. If that happens we can choose a larger compact set $\hat{K} \supset K$, so that $\{z_1,\dots,z_n\} \cap \partial\hat{K} = \phi$. Since the entries $f_{\pm}(t,s,\lambda)$ are holomorphic in K, they take their maximum at the boundary ∂K. The coefficients $a_i(\lambda)$ and the $\nu_i^{\pm}(\lambda)$ are continuous on ∂K and therefore

$$\max_{\lambda\in K} |f_{\pm}(t,s,\lambda)| \leq \tag{4.18}$$

$$\leq r_{\pm} \cdot s^{\alpha} \max_{\substack{\lambda\in\partial K \\ j=1(1)r_{\pm}}} |a_j(\lambda)| \max_{0\leq i\leq r_{\pm}} \left(\exp\left(\frac{c_i^{\pm}}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})\right) \cdot |t^{\alpha+1} - s^{\alpha+1}|^i\right)$$

where $c_i^+ = \min_{\lambda\in\partial K} \operatorname{Re} \nu_i^+(\lambda)$, $c_i^- = \max_{\lambda\in\partial K} \operatorname{Re} \nu_i^-(\lambda)$ hold. Therefore we get

$$\text{(a)} \qquad \max_{\lambda\in K} \|F_+(t,s,\lambda)\| \leq c_1(K) s^{\alpha} \max_{0\leq i\leq r_+} \left(\exp\left(\frac{c_+}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})\right)(s^{\alpha+1} - t^{\alpha+1})^i\right) \tag{4.19}$$

and

$$\text{(b)} \qquad \max_{\lambda\in K} \|F_-(t,s,\lambda)\| \leq \text{const.}\ s^{\alpha} \max_{0\leq i\leq r_-} \left(\exp\left(\frac{c_-}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})\right)(t^{\alpha+1} - s^{\alpha+1})^i\right) \tag{4.19}$$

will $0 < c_+ = \min_{i=1(1)r_+} c_i^+$, $0 > c_- = \max_{i=1(1)r_-} c_i^-$ and $0 \leq i \leq n$ hold.

Using (4.10) and the estimates derived in Markowich (1980b) we get

$$\max_{\lambda\in K} \|H_{\lambda}\|_{[\delta,\infty]} \leq C(K) \tag{4.20}$$

where $C(K)$ is independent of δ.

We rewrite (4.1) as

$$y' = t^{\alpha}A(\infty,\lambda)y + t^{\alpha}(A(t,\lambda) - A(\infty,\lambda))y \tag{4.21}$$

setting

$$G(t,\lambda) = A(t,\lambda) - A(\infty,\lambda) \tag{4.22}$$

We get from (4.21)

$$y(t) = \phi(t,\lambda)V(\lambda)\eta + (H_{\lambda}G(\cdot,\lambda)y)(t), \quad \eta \in \mathbf{C}^{r_-} \tag{4.23}$$

where the columns of the $(n \times r_-)$-matrix $V(\lambda)$, which are holomorphic in Λ°, span the range of $P_-(\lambda)$. The assumptions on $A(t,\lambda)$ guarantee that there is a $\delta > 1$, $\delta = \delta(K)$, such that

$$\|G(\cdot,\lambda)\|_{[\delta,\infty]} < \frac{1}{2C(K)} \quad \text{for all} \quad \lambda \in K \tag{4.24}$$

where K is any compact subset of Λ° and $C(K)$ is as of (4.20). Therefore

$$\max_{\lambda \in K} \|H_{\lambda}G(\cdot,\lambda)\|_{[\delta,\infty]} < \frac{1}{2}. \tag{4.25}$$

This implies that $I - H_{\lambda}G(\cdot,\lambda) : C([\delta,\infty]) \to C([\delta,\infty])$ is nonsingular for all $\lambda \in K$ and

$$y = (I - H_{\lambda}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)V(\lambda)\eta, \quad \eta \in \mathbf{C}^{r_-} \tag{4.26}$$

holds. y is defined for $t \in [\delta,\infty]$ and all $\lambda \in K$. The series expansion for the $n \times r_-$ matrix

$$\psi_-(t,\lambda) = ((I - H_{\lambda}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)V(\lambda))(t) \tag{4.27}$$

is given by

$$\psi_-(\cdot,\lambda) = \sum_{i=0}^{\infty} (H_{\lambda}G(\cdot,\lambda))^{i}\phi(\cdot,\lambda)V(\lambda) \in C([\delta,\infty]), \quad \lambda \in K \tag{4.28}$$

The partial sums $\psi_-^{(k)}(t,\lambda)$ of this series are holomorphic in $\lambda \in \mathring{K}$ for all fixed $t \in [\delta,\infty]$ and because of (4.25) we get

$$\|\psi_-^{(k)}(t)\| \le \left(\sum_{i=0}^{k} \frac{1}{2}\right) \max_{\lambda \in K} \|\phi(\cdot,\lambda)V(\lambda)\|_{[\delta,\infty]} \tag{4.29}$$

$\phi(t,\lambda)V(\lambda)$ is holomorphic in $\Lambda^\circ \supset K$ and therefore the partial sums are uniformly bounded on K and so $\psi_-(t,\lambda)$ is holomorphic in $\lambda \in \mathring{K}$ for all fixed $t \in [\delta,\infty]$. By continuation $\psi_-(t,\lambda)$ is holomorphic in $\lambda \in \mathring{K}$ for all fixed $t \in [1,\infty]$.

Inserting into the boundary condition (4.2) gives the finite dimensional eigenvalue problem

$$F(\lambda)\eta = B(\lambda)\psi_-(1,\lambda)\eta = 0, \quad \eta \in \mathbf{C}^{r_-} \tag{4.30}$$

where $B(\lambda)$ is assumed to a $r_- \times n$ matrix. Given now any compact subset $\theta \subset \Omega$ we choose Λ, K such that $\Lambda^\circ \supset K$, $\mathring{K} \supset \theta$. So $F(\lambda)$ is holomorphic in θ and Theorem 3.1 holds for the problem (4.1), (4.2), (4.3). Therefore, excluding the trivial case, all eigenvalues in Ω are isolated and the dimension of the nullspace is between 1 and r_-. Now we prove the asymptotic estimate for $\psi_-(t,\lambda)$:

$$\max_{\lambda \in \theta} \|\psi_-(t,\lambda)\| \le \text{const.} \exp\left(\left(\max_{\substack{\lambda \in \partial\theta \\ i=1(1)r_-}} \operatorname{Re} \nu_i^-(\lambda) + \varepsilon\right) \frac{t^{\alpha+1}}{\alpha+1}\right) \tag{4.31}$$

where $\nu_i^-(\lambda)$ are the eigenvalues of $A(\infty,\lambda)$ with real part less than zero and $\varepsilon > 0$ is sufficiently small so that the exponent has negative sign.

If there is one (or more) of the singularities $\{z_1,\ldots,z_N\}$ of $E(\lambda)$ on the boundary $\partial\theta$, we take a larger set θ_1 such that $\theta \subset \theta_1$, $\theta_1 \subset K$ and $\{z_1,\ldots,z_N\} \cap \partial\theta_1 = \phi$. Then we derive as in (4.18)

$$\max_{\lambda \in \theta} \|\phi(t,\lambda)\| \le \text{const.} \exp\left(\left(\max_{\substack{\lambda \in \partial\theta \\ i=1(1)r_-}} \operatorname{Re} \nu_i^-(\lambda) + \varepsilon\right) \cdot \frac{t^{\alpha+1}}{\alpha+1}\right) \tag{4.32}$$

The necessity to add $\varepsilon > 0$ in the exponent comes from the possibility that θ might have to be changed to θ_1 described above and from the possible occurence of powers of $|t^{\alpha+1} - s^{\alpha+1}|$. A sufficiently small change and the continuity of the eigenvalues assure that ε is sufficiently small.

$$\max_{\lambda\in\theta} \|(H_\lambda G(\cdot,\lambda)\phi(\cdot,\lambda))(t)\| \le$$

$$\le \max_{\lambda\in\theta} \|G(\cdot,\lambda)\|_{[\delta,\infty]} \Big(\int_t^\infty \max_{\lambda\in\theta} \|F_+(t,s,\lambda)\| \max_{\lambda\in\theta} \|\phi(s,\lambda)\| ds + \tag{4.33}$$

$$+ \int_\delta^t \max_{\lambda\in\theta} \|F_-(t,s,\lambda)\| \max_{\lambda\in\theta} \|\phi(s,\lambda)\| ds \Big)$$

Now (4.19), (4.32) can be used for the estimation of the righthand side of (4.39). θ has to be substituted for K in the definition of c_i^+, c_i^-. Since $\varepsilon > 0$ the estimate given in Markowich (1980b), Chapter 1, Theorem 2.3, can be used and

$$\max_{\lambda\in\theta} \|(H_\lambda G(\cdot,\lambda)\phi(\cdot,\lambda)(t)\| \le \tag{4.34}$$

$$\le \text{const.} \max_{\lambda\in\theta} \|G(\cdot,\lambda)\|_{[\delta,\infty]} \exp\Big(\Big(\max_{\substack{\lambda\in\partial\theta \\ i=1(1)r_-}} \operatorname{Re} \nu_i^-(\lambda) + \varepsilon\Big) \frac{t^{\alpha+1}}{\alpha+1}\Big)\Big)$$

follows. Repeated use of (4.34) and (4.28) gives (4.31)

As in Chapter 3 we investigate the approximating 'finite' eigenvalue problems:

$$x_T' = t^\alpha A(t,\lambda)x_T, \quad 1 \le t \le T, \quad T >> 1 \tag{4.35}$$

$$B(\lambda)x_T(1) = 0 \tag{4.36}$$

$$S(\lambda)x_T(T) = 0 \tag{4.37}$$

where $S(\lambda)$ is a suitably chosen $r_+ \times n$ matrix whose entries are holomorphic in Ω. Rewriting (4.35) as

$$x_T' = t^\alpha A(\infty,\lambda)x_T + t^\alpha G(t,\lambda)x_T, \quad 1 \le t \le T \tag{4.38}$$

where $G(t,\lambda)$ is defined as of (4.22), we set

$$x_T = \phi(t,\lambda)V(\lambda)\eta_- + \phi(t,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\eta_+ + (H_{\lambda;T}G(\cdot,\lambda)x_T)(t) \tag{4.39}$$

where the columns of the $n \times r_+$ matrix $W(\lambda)$, which can be chosen holomorphic in Λ^o, span the range of $P_+(\lambda)$ and $H_{\lambda,T} : C([\delta,T]) \to C([\delta,T])$ is defined as

$$H_{\lambda,T}g = H_\lambda g_T \tag{4.40}$$

for $g \in C([\delta,T])$, $1 \le \delta \le T$ and

$$g_T(t) = \begin{cases} g(t), & \delta \leq t \leq T \\ g(T), & t \geq T \end{cases} \tag{4.41}$$

has been set.

Given a fixed eigenvalue $\lambda = \mu \in \Omega$ of (4.1), (4.2), (4.3), we take a compact subset $K \subset \Lambda^\circ$ with $\mu \in K$ and conclude from (4.20),

$$\max_{\lambda \in K} \|H_{\lambda,T}\|_{[\delta,T]} \leq \max_{\lambda \in K} \|H_\lambda\|_{[\delta,\infty]} \leq C(K) \tag{4.42}$$

and therefore there is a fixed $\delta = \delta(K) \geq 1$ such that

$$\max_{\lambda \in K} \|H_{\lambda,T}G(\cdot,\lambda)\|_{[\delta,T]} \leq \frac{1}{2} \tag{4.43}$$

holds so that $(I - H_{\lambda,T}G(\cdot,\lambda))^{-1}$ exists for all $\lambda \in K$ as an operator on $C([\delta,T])$. We get from (4.39);

$$x_T = (I-H_{\lambda,T}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)V(\lambda)\eta_- + (I-H_{\lambda,T}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\eta_+ \tag{4.44}$$

on $[\delta,\infty]$. The analyticity of

(a) $$_T\psi_-(t,\lambda) = ((I - H_{\lambda,T}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)V(\lambda))(t) \tag{4.45}$$

(b) $$_T\psi_+(t,\lambda) = ((I - H_{\lambda,T}G(\cdot,\lambda))^{-1}\phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda))(t) \qquad (4.45)$$

in λ for $t \in [\delta,T]$ follows as the analyticity of $\psi_-(t,\lambda)$.

The $n \times r_-$ matrix $_T\psi_-$ resp. the $n \times r_+$ matrix $_T\psi_+$ fulfill the equations

(a) $$_T\psi_-(\cdot,\lambda) - (H_{\lambda,T}G(\cdot,\lambda)_T\psi_-(\cdot,\lambda))(\cdot) = \phi(\cdot,\lambda)V(\lambda) \tag{4.46}$$

(b) $$_T\psi_+(\cdot,\lambda) - (H_{\lambda,T}G(\cdot,\lambda)_T\psi_+(\cdot,\lambda))(\cdot) = \phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda) \qquad (4.46)$$

Similar to de Hooq and Weiss (1980a) we derive some properties of $_T\psi_-$, $_T\psi_+$. From (4.27) and (4.46)(a) we get for $_TZ_- = {_T\psi_-} - \psi$

$$_TZ_- = H_{\lambda,T}G(\cdot,\lambda)_TZ_- + (H_{\lambda,T}G(\cdot,\lambda)\psi_- - H_\lambda G(\cdot,\lambda)\psi_-) \tag{4.47}$$

and therefore we get by regarding $G(\cdot,\lambda)\psi_-(\cdot,\lambda) \in C([\delta,T])$

$$ {}_T\psi_-(\cdot,\lambda) - \psi_-(\cdot,\lambda) = (I - H_{\lambda,T}G(\cdot,\lambda))^{-1}(H_{\lambda,T} - H_\lambda)G(\cdot,\lambda)\psi_-(\cdot,\lambda) \in C([\delta,T]) \quad (4.48) $$

Obviously for $g \in C([\delta.T])$ and $t \in [\delta,T]$

$$ ((H_{\lambda,T} - H_\lambda)g)(t) = \phi(t,\lambda)\int_\infty^T P_+(\lambda)\phi^{-1}(s,\lambda)s^\alpha(g(T) - g(s))ds = \quad (4.49) $$

$$ = \phi(t,\lambda)\phi^{-1}(T,\lambda)P_+(\lambda)\phi(T,\lambda)\int_\infty^T P_+(\lambda)\phi^{-1}(s,\lambda)s^\alpha(g(T) - g(s))ds = $$

$$ = \phi(t,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\gamma(g,T) $$

where $\gamma(g,T) \in \mathbb{C}^{r_+}$ holds and therefore

$$ ((H_{\lambda,T} - H_\lambda)G(\cdot,\lambda)\psi_-(\cdot,\lambda))(t) = \phi(t,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\Gamma_T $$

is fulfilled. Γ_T is a $r_+ \times r_-$ matrix. From (4.47) and (4.48) we derive

$$ {}_T\psi_-(\cdot,\lambda) = \psi_-(\cdot,\lambda) + {}_T\psi_+(\cdot,\lambda)\Gamma_T \quad (4.50) $$

Therefore the matrix $[\psi_-(t,\lambda), {}_T\psi_+(t,\lambda)]$ has rank n for all $t \in [\delta,T]$ and is a fundamental matrix of (4.35).

Instead of (4.44) we can write the general solution of (4.35) as

$$ x_T = \psi_-(t,\lambda)\eta_- + {}_T\psi_+(t,\lambda)\eta_+ \quad (4.51) $$

For the following we need an estimate for ${}_T\psi_+(\cdot,\lambda)$. From (4.45)(b) we derive

$$ \max_{\lambda\in\theta} \| {}_T\psi_+(\cdot,\lambda) - \phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\|_{[\delta,T]} \le $$

$$ \le \text{const.} \max_{\lambda\in\theta} \|G(\cdot,\lambda)\phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\|_{[\delta,T]} \quad (4.52) $$

Using similar analyticity arguments as above it is easy to check that the right hand side of (4.52) can be estimated by

$$ w(T,\kappa) = \max_{t\in[\delta,T]} \left(\max_{\lambda\in\theta} \|G(t,\lambda)\| \exp\left(\frac{\kappa - \varepsilon}{\alpha + 1}(t^{\alpha+1} - T^{\alpha+1})\right)\right) \quad (4.53) $$

where $\kappa \equiv \kappa(\theta) = \min\limits_{\substack{\lambda \in \partial\theta \\ i=1(1)r_+}} \operatorname{Re} \nu_i^+(\lambda)$ and $\varepsilon > 0$ sufficiently small.

Obviously $\lim\limits_{T\to\infty} w(T,\kappa) = 0$ and we get after continuation to $[1,T]$

$$\lim_{T\to\infty} \| {}_T\psi_+(\cdot,\lambda) - \phi(\cdot,\lambda)\phi^{-1}(T,\lambda)W(\lambda)\|_{[1,T]} = 0 \tag{4.54}$$

uniformly for $\lambda \in \theta$.

Now we evaluate (4.51) at the boundaries $t = 1,T$ and use the boundary conditions (4.36), (4.37) getting the n-dimensional nonlinear eigenvalue problem

$$F(\lambda,T)\binom{\eta_-}{\eta_+} = \begin{bmatrix} B(\lambda)\psi_-(1,\lambda) & B(\lambda)_T\psi_+(1,\lambda) \\ S(\lambda)\psi_-(T,\lambda) & S(\lambda)_T\psi_+(T,\lambda) \end{bmatrix} \binom{\eta_-}{\eta_+} = 0 \tag{4.55}$$

Interpreting (4.55) as perturbed eigenvalue problem of

$$\begin{bmatrix} B(\lambda)\psi_-(1,\lambda) & 0 \\ 0 & S(\lambda)W(\lambda) + o(T,\lambda) \end{bmatrix} \binom{\eta}{\xi} = 0 \tag{4.56}$$

we get with $F(\lambda) = B(\lambda)\psi_-(1,\lambda)$:

$$\begin{aligned} \det F(\lambda,T) &= \det F(\lambda)(\det S(\lambda)W(\lambda) + o(T,\lambda)) + \\ &+ O(\|{}_T\psi_+(1,\lambda)\| \cdot \|S(\lambda)\psi_-(T,\lambda)\|) \end{aligned} \tag{4.57}$$

where

$$|o(T,\lambda)| \to 0 \quad \text{as} \quad T \to \infty \tag{4.58}$$

uniformly for $\lambda \in \theta$.

Assuming that $S(\lambda)W(\lambda)$ is nonsingular we get by cancelling $\det S(\lambda)W(\lambda) + o(T,\lambda)$ and by applying the perturbation arguments of Chapter 2:

<u>Theorem 4.1</u>. Let the $r_+ \times n$ matrix $S(\lambda)$ be holomorphic for $\lambda \in \Omega$ and assume that $\det S(\lambda)W(\lambda) \neq 0$ in Ω. Let $\mu \in \Omega$ be an eigenvalue of (4.1), (4.2), (4.3) of order s, i.e. $\det F(\lambda) = \det B(\lambda)\psi_-(1,\lambda)$ has a zero of order s at $\lambda = \mu$. Then there are exactly s eigenvalues $\mu_T^1,\ldots,\mu_T^s$ (counting multiplicities of the zeros of $\det F(\lambda,T)$) for T sufficiently large in a sufficiently small neighbourhood $\tilde{S}_\mu$ of μ and

$$\max_{i=1(1)s} |\mu_T^i - \mu| \leq \text{const.}\left(w(T,\kappa(\bar{S}_\mu))\right)^{\frac{1}{s}} \exp\left((\nu_-(\mu) + \varepsilon)\,\frac{T^{\alpha+1}}{s(\alpha+1)}\right) \tag{4.59}$$

where $\nu_-(\mu) = \max_{i=1(1)r_-} \operatorname{Re} \nu_i^-(\mu)$ and $w(T,\kappa)$ is defined in (4.53) and $\varepsilon > 0$ is sufficiently small.

$$|\hat{\mu}_T - \mu| \leq \text{const. } w(T,\kappa(\bar{S}_\mu)) \exp\left((\nu_-(\mu) + \varepsilon)\,\frac{T^{\alpha+1}}{\alpha+1}\right) \tag{4.60}$$

$\hat{\mu}_T = \frac{1}{s}\sum_{i=1}^{s} \mu_T^i$ has been set. Let x_T be an eigenfunction belonging to one of the μ_T^i's. Then

$$\inf_{y \in N_\mu} \|x_T - y\|_{[1,T]} \leq \text{const. } \left(w(T,\kappa(\bar{S}_\mu))\right)^{\frac{1}{s}} \exp\left((\nu_-(\mu) + \varepsilon)\,\frac{T^{\alpha+1}}{s(\alpha+1)}\right) \tag{4.61}$$

holds where N_μ denotes the nullspace of (4.1), (4.2), (4.3) for $\lambda = \mu$.

These convergence results are the extension of the convergence results for linear eigenvalue problems given in Markowich (1980). The orders of convergence obtained there hold without any change for nonlinear problems.

A possible choice for $S(\lambda)$ is given by (3.21), i.e. the rows of the holomorphic $r_+ \times n$ matrix $S(\lambda) = S_p(\lambda)$ span the range of $(P_+(\lambda))^T$ (the superscript T denotes transposition). This choice reproduces eigenvalues and eigenvectors exactly in the case that A does not depend on t. However, in the general case this does not hold anymore but in some important cases the asymptotic boundary condition $S_p(\lambda)x_T(T) = 0$ implies a faster order of convergence than given in Theorem 4.1. Assume that $A(t,\lambda)$ decays algebraically or exponentially:

$$A(t,\lambda) = A(\lambda) + O(t^\gamma e^{-a(t)}) \quad \text{for} \quad t \to \infty \tag{4.62}$$

uniformly in compact subset $K \subset \Omega$ where $\gamma \in \mathbb{R}$ and $a(t) \geq 0$ is a real function such that $t^\gamma e^{-a(t)} \to 0$ as $t \to \infty$. Then since $S_p(\lambda)\phi(T,\lambda)V(\lambda) \equiv 0$ we get from (4.46a)

$$\|S_p(\lambda)\psi_-(T,\lambda)\| = \|S_p(\lambda)(H_\lambda G(\cdot,\lambda)\psi_-(\cdot,\lambda))(T)\| \leq$$

$$\leq \text{const. } T^\gamma e^{-a(T)} \exp\left(\left(\max_{i=1(1)r_-} \operatorname{Re} \nu_i^-(\lambda) + \varepsilon\right)\frac{T^{\alpha+1}}{\alpha+1}\right)$$

This follows from the estimates given in Markowich (1980b) applied to (4.27). In this case the right hand side of the estimate (4.58), (4.61) given in Theorem 4.1 can be multiplied by $(T^\gamma e^{-a(T)})^{\frac{1}{s}}$ and the right hand side of (4.60) can be multiplied by $T^\gamma e^{-a(T)}$.

Now we consider the case of simple eigenvalues of (4.1), (4.2), (4.3). Since we only have defined simple eigenvalues for nonlinear finite-dimensional eigenvalue problems we give

Definition 4.1: An eigenvalue $\mu \in \Omega$ of (4.1), (4.2), (4.3) is called simple if the corresponding nullspace is one dimensional, say it is spanned by the normed vector y, and if the problem

$$v' - t^\alpha A(t,\mu)v = t^\alpha A_\lambda(t,\mu)y(t) \tag{4.63}$$

$$B(\mu)v(1) + B_\lambda(\mu)y(1) = 0 \tag{4.64}$$

$$v \in C([1,\infty]) \tag{4.65}$$

has no solution. Now we show

Theorem 4.2. The eigenvalue μ of (4.1), (4.2), (4.3) is simple iff μ is a first order zero of $\det F(\lambda) = 0$.

Proof: Since y is an eigenvector corresponding to the eigenvalue $\mu \in \Omega$

$$y(t,\mu) = \psi_-(t,\mu)\xi$$

for some $\xi \in \mathbb{C}^{r_-}$ holds. Obviously

$$y_\lambda(t,\mu) = \frac{d}{d\lambda}\psi_-(t,\mu)\xi$$

is a particular solution of (4.63), (4.65) therefore the general solution of (4.63) is

$$v(t) = \psi_-(t,\mu)\beta + y_\lambda(t,\mu), \quad \beta \in \mathbb{C}^{r_-}$$

Inserting into (4.64) gives

$$B(\mu)\psi_-(1,\mu)\beta = -(B(\mu)\frac{d}{d\lambda}\psi_-(1,\mu) + B_\lambda(\mu)\psi_-(1,\mu))\xi$$

or

$$F(\mu)\beta = -F_\lambda(\mu)\xi .$$

This equation is unsolvable (for β) iff the generalized linear eigenvalue problem $(F(\mu) + \kappa F_\lambda(\mu)) = 0$ has $\kappa = 0$ is an eigenvalue with geometric and algebraic multiplicity 1. This holds iff $\det F(\lambda) = 0$ has a first order zero at $\lambda = \mu$.

Now we show that approximations for an eigenvalue-eigenvector pair (λ, y) can be computed - in the case that λ is simple - as solutions to nonlinear 'finite' two point boundary value problems. Lentini and Keller (1980) did computation pursuing this way.

We set, assuming that $\mu \in \Omega$ is simple

$$y_{n+1} = \mu, \quad z = (y_1, \ldots, y_n, y_{n+1})^T, \quad \sum_{i=1}^{n} y_i^2(1) = 1 \tag{4.66}$$

and get from (4.1), (4.2), (4.3)

$$z' = t^\alpha \begin{bmatrix} A(t, z_{n+1}) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \\ 0 \end{bmatrix} = t^\alpha f(t,z), \quad 1 < t < \infty \tag{4.67}$$

$$\begin{bmatrix} B(z_{n+1}(1)) \begin{pmatrix} z_1(1) \\ \vdots \\ z_n(1) \end{pmatrix} \\ \sum_{i=1}^{n} z_i^2(1) - 1 \end{bmatrix} = b(z(1)) = 0 \tag{4.68}$$

$$z \in C([1,\infty]) \tag{4.69}$$

The condition $\sum_{i=1}^{n} y_i^2(1) = 1$ shall sort one eigenfunction $y \not\equiv 0$ out of the one dimensional eigenspace.

(4.67), (4.68), (4.69) is a singular two point boundary value problem as described by de Hoog and Weiss (1980a,b), Markowich (1980a,b,c) and Lentini and Keller (1980). Since all eigenvalues in Ω are isolated and since μ is simple the solution $z = (y_1, \ldots, y_n, \mu)^T$ of (4.67), (4.68), (4.69) is locally unique. Now we will show that z is isolated, i.e. the linearized problem, has only the zero-solution. We get for the linearized problem with $u = (u_1, \ldots, u_n, u_{n+1})^T$

$$u' = t^\alpha \begin{bmatrix} A(t,\mu) & A_\lambda(t,\mu)y(t) \\ 0 & 0 \end{bmatrix} u \tag{4.70}$$

$$\begin{bmatrix} B(\mu) & B_\lambda(\mu)y(1) \\ 2y(1)^T & 0 \end{bmatrix} u(1) = 0 \tag{4.71}$$

$$u \in C([1,\infty]) \tag{4.72}$$

Setting $v = (u_1,\ldots,u_n)^T$ we derive

(a) $$v' - t^\alpha A(t,\mu)v = u_{n+1}t^\alpha A_\lambda(t,\mu)v \tag{4.73}$$

(b) $$u_{n+1} = \text{const.} \tag{4.73}$$

(a) $$B(\mu)v(1) + u_{n+1}(1)B_\lambda(\mu)y(1) = 0 \tag{4.74}$$

(b) $$y(1)^T v(1) = 0 \tag{4.74}$$

$$v \in C([1,\infty]) \ . \tag{4.75}$$

Because of Definition 4.1 the problem (4.73), (4.74), (4.75) has no solution unless $u_{n+1} \neq 0$. If $u_{n+1} \neq 0$ then v has to be an eigenfunction of (4.1), (4.2), (4.3), therefore $v = cy$ for some constant c. (4.74)(b) gives $c = 0$, such that $u \equiv 0$ follows as the unique solution of (4.70), (4.71), (4.72). Therefore, we conclude from Markowich (1980b) that the infinite problem (4.67), (4.68), (4.69) can be approximated by finite interval problems of the form

$$w_T' = t^\alpha f(t,w_T), \quad 1 \leq t \leq T \tag{4.76}$$

$$b(w_T(1)) = 0 \tag{4.77}$$

$$S(w_T(T)) = 0 \tag{4.78}$$

where $w_T = (w_T^1,\ldots,w_T^n,w_T^{n+1})^T$ and $(w_T^1,\ldots,w_T^n)$ is the approximation to the eigenvector y and w_T^{n+1} is the approximation to λ. The choice of $S : \mathbb{R}^{n+1} \to \mathbb{R}^{r_+}$ is explained in Markowich (1980b). The analysis given here shows that we can take

$$S(w_T(T)) = S_p(w_T^{n+1}(T)) \cdot \begin{pmatrix} w_T^1(T) \\ \vdots \\ w_T^n(T) \end{pmatrix} \tag{4.79}$$

The superscript T denotes transposition. Markowich (1980b) showed that the solution w_T is locally (around z) unique for T sufficiently large and that

$$\|z - w_T\|_{[1,T]} \leq \text{const.}\ \|S(z(T))\| = \text{const.}\ \|S_p(\mu)y(T)\| \tag{4.80}$$

holds and therefore we get the order of convergence given in Theorem (4.1) with $s = 1$, because the boundary condition (4.78) is equivalent to (3.21).

If (4.63) holds the order of convergence is

$$|\mu_T - \mu| \leq \text{const.}\ T^{\gamma}\exp\Big(\big(\max_{i=1(1)r_-} \operatorname{Re}\ \nu_i(\lambda) + \varepsilon\big) - a(T)\Big)$$

and the same holds for the normed eigenvectors.

5. CASE STUDIES.

The first problem we treat is the so called radial Schrödinger equation of the Kepler-problem (see Jürgens-Rellich (1976), Chapter 3, par. 9) which is given by

$$-u'' + \{(1 + \ell)\ell r^{-2} - 2cr^{-1}\}u = -\lambda u, \quad 1 \leq r < \infty \tag{5.1}$$

where $\ell \in \mathbb{N}_0$, $c \in \mathbb{R}$ holds.

The transformation

$$y = (y_1, y_2)^T = (u, u')^T \tag{5.2}$$

takes (5.1) into the system

$$y' = \underbrace{\begin{bmatrix} 0 & 1 \\ (1 + \ell)\ell r^{-2} - 2cr^{-1} + \lambda & 0 \end{bmatrix}}_{A(r,\lambda)} y, \quad 1 \leq r < \infty \tag{5.3}$$

such that

$$A(\infty, \lambda) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} \tag{5.4}$$

holds.

The eigenvalue problem (5.3) is linear but we will construct the nonlinear (in λ) asymptotic boundary condition $S_p(\lambda)$.

The Jordan form $J(\infty,\lambda)$ of $A(\infty,\lambda)$ is

$$J(\infty,\lambda) = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix},\ r_+ = 1,\ r_- = 1,\ D_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ D_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.5}$$

and therefore the set $\Omega = \{\lambda \in \mathbb{C} | \mathrm{Re}(\sqrt{\lambda}\,) \neq 0\}$ is given by

$$\Omega = \mathbb{C} - \{\lambda | \mathrm{Re}\ \lambda \leq 0\}\ . \tag{5.6}$$

With appropriate boundary conditions of the form

$$a_1 y_1(1) + a_2 y_2(1) = 0 \tag{5.7}$$

we conclude from Chapter 4 that every eigenvalue $\lambda \in \Omega$ of (5.3), (5.7) is isolated and that the dimension of the nullspace equals 1.

A complete analysis of the problem is given in Jürgens and Rellich (1976). They show that there is an infinite sequence of eigenvalues $\lambda^{(n)}$

$$\lambda^{(n)} = c^2(\ell + 1 + n)^{-2} \quad \forall\ n \in \mathbb{N}_0 \tag{5.8}$$

and the eigenfunctions $y^{(n)}$ are given by

$$y^{(n)}(r) \equiv \exp\left(-\sqrt{\lambda^{(n)}}\ r\right) r^{\ell+1} p_n(r) = \exp\left(-\sqrt{\lambda^{(n)}}\ r\right) r^{\ell+n+1}(1 + O(r^{-1})) \tag{5.9}$$

because $p_n(r)$ is a polynomial in r of degree n. They assumed that $a_1 = \sin\alpha$, $a_2 = \cos\alpha$ with $\alpha \in [0,\infty)$.

A straightforward calculation gives for $\lambda \in \Omega$

$$E(\lambda) = \begin{bmatrix} \sqrt{\lambda} & -\sqrt{\lambda} \\ \lambda & \lambda \end{bmatrix}, \quad E^{-1}(\lambda) = \begin{bmatrix} \frac{1}{2\sqrt{\lambda}} & \frac{1}{2\lambda} \\ \frac{1}{-2\sqrt{\lambda}} & \frac{1}{2\lambda} \end{bmatrix}, \quad P_+(\lambda) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{\lambda}} \\ \frac{\sqrt{\lambda}}{2} & \frac{1}{2} \end{bmatrix} \tag{5.10}$$

and therefore, since $P_+(\lambda) = E(\lambda) D_+ E^{-1}(\lambda)$

$$S_p(\lambda) = [\sqrt{\lambda}, 1] \tag{5.11}$$

holds.

The approximating problems have the form

$$x_R' = A(r,\lambda)x_R, \quad 1 \leq r \leq R, \quad R >> 1 \tag{5.12}$$

$$[a_1, a_2]x_R(1) = 0 \tag{5.13}$$

$$[\sqrt{\lambda}, 1]x_R(R) = 0 \Leftrightarrow \sqrt{\lambda}\ x_R^1(R) + x_R^2(R) = 0, \quad x_R = (x_R^1, x_R^2)^T \tag{5.14}$$

Since $A(r,\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} + 0(r^{-1})$ and since $w(R,\mathrm{Re}\sqrt{\lambda}\ ) \leq$ const. R^{-1} the convergence analysis given in Chapter 4 shows that

$$\max\left(|\lambda^{(n)} - \lambda_R^{(n)}|,\ \|y^{(n)} - x_R^{(n)}\|_{[1,R]}\right) \leq ce^{-\sqrt{\lambda^{(n)}}R}\ R^{\ell+n-1} \tag{5.15}$$

under the assumption that the $\lambda^{(n)}$'s are simple. The $\lambda_R^{(n)}$'s resp. $x_R^{(n)}$'s are the eigenvalues resp. eigenvectors of (5.12), (5.13), (5.14) which approximate $\lambda^{(n)}$ resp. $y^{(n)}$.

(5.15) follows by using $\psi_-(r,\lambda^{(n)}) = [y^{(n)}(r), \frac{d}{dr}\, y^{(n)}(r)]^T$. Therefore the estimate (4.81), where ε appears in the exponent, can be improved using (4.57). From Markowich (1980a) we conclude that every boundary condition $Sx_R(R) = 0$ where S is independent of λ and where

$$SE(\lambda)\binom{1}{0} \neq 0 \quad \text{in} \quad \Omega \tag{5.16}$$

holds, leads to convergence of the order

$$\begin{aligned} &\max\left(|\lambda^{(n)} - \lambda_R^{(n)}|, \|y^{(n)} - x_R^{(n)}\|_{[1,R]}\right) \leq \\ &\leq c_1\|y^{(n)}(R)\|w(R,\mathrm{Re}(\sqrt{\lambda}\ )) \leq c\ \exp\left(-\sqrt{\lambda^{(n)}}\ R\right)R^{\ell+n} \end{aligned} \tag{5.17}$$

Setting $S = [s_1,s_2]$, $s_1 s_2 \in C$, (5.16) is fulfilled iff

$$s_1 + s_2\sqrt{\lambda} \neq 0 \tag{5.18}$$

holds. For example the natural boundary condition

$$x_R^1(R) = 0 \tag{5.19}$$

fulfills (5.18) ($s_1 = 1$, $s_2 = 0$) and the order of convergence, given by (5.17) differs only by one power of R from the order of convergence produced by the 'optimal' boundary condition (5.14).

The second problem we deal with is the Orr-Sommerfeld equation (see Ng and Reid (1980)) which governs the stability of a laminar boundary layer in a parallel flow approximation.

$$\frac{1}{iR\alpha}\left(\frac{d^2}{dz^2} - \alpha^2\right)^2\phi - \left\{(U(z) - \lambda)\left(\frac{d}{dz^2} - \alpha^2\right)\phi - U''(z)\phi\right\} = 0 \tag{5.20}$$

with $\alpha > 0$, $R > 0$ is the Reynolds number, $U(z)$ is the velocity distribution fulfilling

$$U(z) = 1 + F(z)e^{-\omega z^2}, \quad \omega > 0, \quad F \in C^2([0,\infty]) \tag{5.21}$$

such that $U(\infty) = 1$, $U''(\infty) = 0$ holds. $\phi(z)e^{i\alpha(x-\lambda t)}$ represents the disturbance stream function. The boundary conditions for the Orr-Sommerfeld problem are,

$$\phi(0) = \phi'(0) = \phi(\infty) = \phi'(\infty) = 0 \tag{5.22}$$

The substitution

$$y = (\phi, \phi', \phi'', \phi''')^T \tag{5.23}$$

gives the linear eigenvalue problem

$$y' = \underbrace{\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_1(z) & 0 & f_2(z) & 0 \end{array}\right] + \lambda \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & b & 0 \end{array}\right]}_{A(z,\lambda)} y, \quad 0 \leq z < \infty \tag{5.24}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] y(0) = 0 \tag{5.25}$$

$$y \in C([0,\infty]) \tag{5.26}$$

where

(a) $$f_1(z) = -(\alpha^4 + i\alpha R(\alpha^2 U(z) + U''(z))) \tag{5.27}$$

(b) $$f_2(z) = 2\alpha^2 + i\alpha RU(z) \tag{5.27}$$

(c) $$a = i\alpha^3 R \qquad (5.27)$$

(d) $$b = -i\alpha R \qquad (5.27)$$

holds. The eigenvalues of $A(\infty,\lambda)$ are

$$\nu_1(\lambda) = \alpha, \nu_2(\lambda) = (\alpha^2 + i\alpha R(1-\lambda))^{1/2}, \nu_3(\lambda) = -\alpha, \nu_4(\lambda) = -(\alpha^2 + i\alpha R(1-\lambda))^{1/2} \qquad (5.28)$$

so that $\text{Re } \nu_1(\lambda)$, $\text{Re } \nu_2(\lambda) > 0$; $\text{Re } \nu_3(\lambda)$, $\text{Re } \nu_4(\lambda) < 0$ for all $\lambda \in \Omega = C - \{\lambda | \text{Re } \lambda = 1, \text{ Im } \lambda \leq -\frac{\alpha}{R}\}$. All eigenvalues $\lambda \in \Omega$ of (5.24), (5.25), (5.26) are isolated and the nullspaces are at most two-dimensional.

The approximating problems have the form

$$x_Z' = A(z,\lambda)x_Z, \quad 0 \leq z \leq Z \qquad (5.28)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_Z(0) = 0 \qquad (5.30)$$

$$S(\lambda)x_Z(Z) = 0$$

where $x_Z = (x_Z^1, x_Z^2, x_Z^3, x_Z^4)$ holds and $\lambda \in \Omega$.

As for the first example we calculate the 'optimal' boundary condition $S(\lambda) = S_p(\lambda)$

$$S_p(\lambda) = \begin{bmatrix} 0 & 1 & \nu_2(\lambda) & \alpha\nu_2(\lambda) \\ 1 & 0 & -(\nu_2(\lambda) + \alpha\nu_2(\lambda) + \alpha^2) & -\alpha\nu_2(\lambda)(\nu_2(\lambda) + \alpha) \end{bmatrix} \qquad (5.32)$$

Since $A(z,\lambda) = A(\lambda) + 0\left(z^2 e^{-\omega z^2}\right)$ holds we get from (4.81) for simple eigenvalues $\lambda = \mu \in \Omega$

$$\max(|\mu - \mu_Z|, \|y - x_Z\|_{[0,Z]}) \leq$$
$$\leq \text{const } Z^2 e^{-\omega Z^2} w(Z, \min(\alpha, \text{Re } \nu_2(\mu))) \exp(\max(-\alpha, \text{Re } \nu_4(\mu) + \varepsilon)Z) \qquad (5.33)$$

where y, x_Z are the normed eigenfunctions. In the most interesting case $\alpha < |\mathrm{Re}\ \nu_4(\lambda)| < 1$ the order of convergence is $Z^2 \exp(-\omega Z^2 - 2(\alpha - \varepsilon)Z)$, $\varepsilon > 0$ sufficiently small, and linear asymptotic boundary conditions, for example

$$x_z^1(Z) = x_z^2(Z) = 0 \tag{5.34}$$

achieve a slower order of convergence, namely $\exp(-2(\alpha - \varepsilon)Z)$ (see Markowich (1980a)).

For this problem the nonlinear asymptotic boundary conditions achieve a much faster order of convergence.

Numerical experiments were performed by Ng and Reid (1980).

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This paper is concerned with nonlinear eigenvalue problems of boundary value problems for ordinary differential equation posed on an infinite interval. It is shown that under certain analyticity assumptions - a domain in the complex plain can be identified, in which all eigenvalues are isolated. An intriguing way to solve such problems is to cut the infinite interval at a finite but large enough point and to impose additional, so called asymptotic boundary conditions at this far end. The obtained eigenvalue problem for the two point boundary value problem on this finite but

(continued)

ABSTRACT (continued)

large interval can be solved by any appropriate code. In this paper suitable asymptotic boundary conditions are devised and the order of convergence, as the length of the interval, on which these approximating problems are posed, converges to infinity, is investigated. Exponential convergence is shown for well posed approximating problems.

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